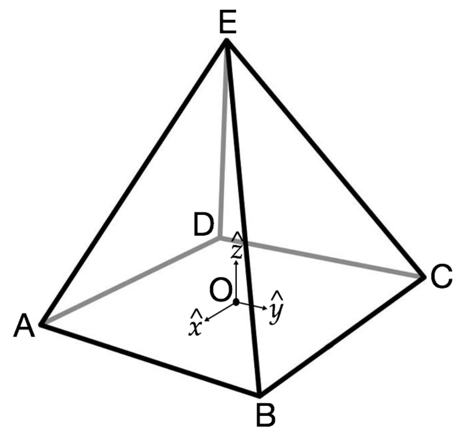
Exam Symmetry in Physics

Date	April 5, 2023
Time	8:30 - 10:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- Use of a calculator is allowed
- All subquestions (a, b, c) of the three exercises have equal weight
- Illegible answers will be not be graded
- Good luck!

Exercise 1

Consider a pyramid with a regular square as base with corners labeled by A, B, C, D and its apex E above the center O of the square base (see figure). Its symmetry group is C_{4v}.



(a) Identify all symmetry transformations that leave this pyramid invariant and divide them into conjugacy classes, using geometrical arguments.

(b) Construct the character table of C_{4v} and explain how the entries are obtained.

(c) Determine the characters of the three-dimensional vector representation of C_{4v} and use them to determine whether the symmetries of the pyramid allow for an invariant vector.

Exercise 2

Consider the group $O(3)$ of real orthogonal 3×3 matrices and its action on a rank-2 tensor σ_{ij} (i, j can take the values 1, 2, and 3).

(a) Show that $\sigma_{ij} = \delta_{ij}$ (the Kronecker delta) is invariant under $O(3)$ transformations.

(b) Show that any $O(3)$-invariant tensor σ_{ij} satisfies $[D^V(g), \sigma] = 0$ for all $g \in O(3)$.

(c) Determine the subgroup of $O(3)$ transformations that leave the tensor

$$\sigma_{ij} = \delta_{ij} + a\delta_{i3}\delta_{j3}$$

invariant, for a nonzero constant a.

Exercise 3

Consider the group $SU(2)$ of unitary 2×2 matrices with determinant equal to 1. Consider its action on the spin states $|s, m_s\rangle$ through the operator

$$U(\theta, \hat{n}) = \exp\left(\frac{i}{\hbar}\theta\, \hat{n} \cdot \vec{S}\right),$$

where $\vec{S}$ denotes the spin operator.

(a) Write down the explicit matrix representation $D^{(\frac{1}{2})}$ for $U(\theta, \hat{n})$ acting on the carrier space of $s = \frac{1}{2}$ states for the specific case $\hat{n} = \hat{z}$.

(b) Explain how the result in (a) shows that states of a spin-$\frac{1}{2}$ particle pick up a minus sign after a 2π rotation and that as a consequence $D^{(\frac{1}{2})}$ is not a representation of $SO(3)$.

(c) Write down the Clebsch-Gordan series for the direct product $D^{(\frac{1}{2})} \otimes D^{(\frac{3}{2})}$ and verify the decomposition by checking the dimensions of the various representations (no need to explicitly indicate the basis transformation).